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BREAK-THROUGH WAVE OVER A DEFORMABLE BOTTOM

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A mathematical model of liquid flow over a bottom being washed out is suggested and investigated.

In the one-dimensional St. Venant approximation with an undeformable bottom the problem of a break-through wave was solved numerically on a computer for the case of a dry channel by Sudobicher [1, 2] and for the case of a wet channel by Gladyshev (see [3] and the bibliography given there). In the case of a constant slope of the bottom and a uniform channel this problem admits an asymptotic solution for $t \to \infty$ (in the most general form this solution is given in [3]).

To calculate nonstationary liquid motion over a bottom being washed out, there is an approach in the literature where a determinate equation (different authors have different equations) for bottom deformation is assigned to St. Venant equations. Up to now, problems for a constant region have been solved [4-6].

Below, in using the above approach, the equation for the bottom deformation is not fixed in advance, but is derived from certain physical considerations. For the first time the problem of a break-through wave (the case of a dry bottom) in a variable region is considered, and a mechanism of detention of particles of the ground that forms the channel bottom is suggested. Thereby, a mathematical interpretation of the notion of "washing-out velocity," well-known in hydraulics, is given.

We consider the traditional approach to describing flows over a deformable bottom, where an equation of channel deformation is added to the St. Venant equations. For a slope we have

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0\,, \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} + g\frac{\partial z_0}{\partial x} = -\frac{ku\,|u|}{h}\,,$$
$$\frac{\partial z_0}{\partial t} + \frac{\partial}{\partial x}Q(u, h) = 0\,. \tag{1}$$

The functions h, u, and z_0 are sought functions. The function Q is determined below. By applying a standard method, we obtain an equation for $dx/dt = \lambda$:

$$\lambda^3 - 2u\lambda^2 + \left(u^2 - gh - g\frac{\partial Q}{\partial u}\right)\lambda - gh\frac{\partial Q}{\partial h} + gu\frac{\partial Q}{\partial u} = 0\,. \tag{2}$$

From physical considerations it is required that Eq. (2) could have the root $\lambda_1 = u$. Substitution of the quantity u in place of λ in Eq. (2) yields $\partial Q/\partial h = -u$. From this it follows that $Q = -uh + f(u)$. For the function $f(u)$ we take the power-law dependence $f(u) = \alpha u^m$, as the most simple. Having divided the left-hand side of Eq. (2), obtained after substitution of Q, by $\lambda - u$, we come to the equation $\lambda^2 - u\lambda - g\alpha m u^{m-1} = 0$, from which

$$\lambda_{2,3} = \frac{u}{2} \pm \sqrt{\left(\frac{u^2}{4} + g\alpha\, m u^{m-1}\right)}\,. \tag{3}$$

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In order to satisfy the condition $\lambda \neq 0$ or $\lambda \neq \infty$ at $u = 0$, it is necessary that $m = 1$. Actually, when $u = 0$ and $m > 1$, we have all three characteristics of zero slope ($\lambda_i = 0$, $i = 1, 2, 3$), i.e., here weak disturbances do not propagate in the liquid at rest. The case $\lambda_{2,3} = \pm\infty$ ($m < 1$) is impossible, since wave processes are described by a hyperbolic equation when the velocities of propagation of weak disturbances are finite. It remains that $m = 1$.

Thus, by purely logical means we come to the formula

$$Q(u, h) = -uh + \alpha u, \tag{4}$$

known in the literature [4] and obtained from experiments and theoretical considerations that differ from those stated above. Of interest is the fact that all three characteristics are related only to u and are independent of h and z_0. The characteristics λ_2 and λ_3 always differ in sign. Consequently, there is no supercritical flow, no stationary breaks exist, and no breaks ahead of a streamlined body are observed.

System of equations (1) and (4) can be investigated by different methods. For example, the expression on the characteristic $dx/dt = u$ at $k = 0$ leads to a simplification, namely, the splitting $z_0 = \alpha \ln h - h - u^2/2g + C$, and relates to $\partial h/\partial t + \partial(uh)/\partial x = 0$ and $\partial u/\partial t + g\alpha/h \cdot \partial h/dx = 0$. Actually, we multiply the first expression of (1) by $-g(\alpha - h)/h$, the second by u, and the third by g with allowance for Eq. (4) and then add them together. As a result, we obtain the relation

$$-\frac{g(\alpha - h)}{h}\left(\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}\right) + u\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + g\left(\frac{\partial z_0}{\partial t} + u\frac{\partial z_0}{\partial x}\right) = -\frac{ku^2|u|}{h}.$$

From this,

$$-\frac{g(\alpha - h)}{h}\,dh + u\,du + g\,dz_0 = -\frac{ku^2|u|}{h}\,dt$$

along $dx/dt = u$ or at $k = 0$

$$-g(\alpha \ln h - h) + \frac{1}{2}u^2 + gz_0 = gC = \text{const}.$$

It is considered that the last equality is satisfied over the entire flow. Thus, we obtain the first relation, and after elimination of z_0 from the second expression of (1), we come to the second relation of the simplified system. By the way, when $m = 1$, the characteristics of the latter $dx/dt = u/2 \pm \sqrt{u^2/4 + g\alpha}$ coincide with Eqs. (3).

If we apply a reduced method of disturbances [3] to the system (1) and (4), we obtain the expression

$$A\frac{\partial h_1}{\partial \tau} + Bh_1\frac{\partial h_1}{\partial \xi} = -C\left(\sqrt{\left(\frac{u_0^2}{4} + g\alpha\right)} - u_0\right)h_1. \tag{5}$$

where $\xi = \varepsilon^{-1}(x - \lambda t)$, $\tau = t$; $h = h_0 + \varepsilon h_1 + \ldots$; $u = u_0 + \varepsilon u_1 + \ldots$; $z_0 = -i(\varepsilon\xi + \lambda t) + \varepsilon z_1 + \ldots$; ε is the fictitious small parameter necessary for reduction (it is assumed in the final formulas that $\varepsilon = 1$); the quantity C is proportional to k. The other parameters are as follows:

$$gih_0 = ku_0^2,\quad \lambda = \frac{u_0}{2} + \sqrt{\left(\frac{u_0^2}{4} + g\alpha\right)},\quad u_1 = \left(\sqrt{\left(\frac{u_0^2}{4} + g\alpha\right)} - \frac{u_0}{2}\right)\frac{h_1}{h_0},$$

$$z_1 = \left[\frac{\left(\frac{u_0}{2} - \sqrt{\left(\frac{u_0^2}{4} + g\alpha\right)}\right)^2}{gh_0} - 1\right]h_1.$$

Equation (5) describes the propagation of a unidirectional wave packet of small but finite amplitude. This packet is observed at large times far from the site of the initial disturbance. If we take $\lambda = u_0/2 - (u_0^2/2 + g\alpha)^{1/2}$, then Eq. (5) describes the propagation of waves to the other side. The constancy of λ shows that the velocities of propagation of the wave packet correspond to the linear theory, while the deformation of the shape of the waves is described by nonlinear equation (5). From the form of the right-hand side of Eq. (5) it follows that instability of the uniform flow $h = h_0$, $u = u_0$, $z_0 = -ix$ occurs when $(u_0^2/4 + g\alpha)^{1/2} - u_0 < 0$, i.e., when $u_0 > (4/3g\alpha)^{1/2}$. We rewrite Eq. (5) in the form

$$A\frac{\partial h_1}{\partial \tau} + Bh_1\frac{\partial h_1}{\partial \xi} + Eh_1 = 0\,. \tag{5a}$$

Expression (5a), called the equation of simple waves, is investigated in the literature for smooth solutions in the case where a gradient catastrophe (approach of the derivatives to infinity) is formed. We do not dwell on this case here.

Equation (5a) is a model equation for system (1). We consider a problem with a discontinuity for Eq. (5a) that is a model problem for a break-through wave over a wet bottom. To do this, we set the initial condition $h_1(\xi, 0) = -E\xi/B$, $0 < \xi < 1$ and the boundary condition $h_1(0, \tau) = 0$. According to the standard method, the condition on the discontinuity has the form $D[Ah_1] = [Bh_1^2/2]$, where $[f] = f^+ - f^-$. We assume that $h_1^- = 0$. Then $D = d\xi/d\tau = Bh_1^+/2A$. The solution $h_1(\xi, \tau) = -E\xi/B$ of (5a) satisfies both the initial and boundary conditions. As a result, we have $d\xi/d\tau = -E\xi/2A$. From this we obtain the equation for the discontinuity line $\xi = \exp(-E\tau/2A)$. The final solution takes the form

$$h_1(\xi, \tau) = \begin{cases} 0\,, & \xi < 0\,,\ \xi > \exp(-E\tau/2A) \\ -E\xi/B\,, & 0 < \xi < \exp(-E\tau/2A) \end{cases}.$$

When $E < 0$, the discontinuity increases in amplitude and extends toward an increase in ξ.

Now we consider a flow with a constant velocity of the leading edge. For simplification we substitute $z_0 = -ix + \tilde{z}_0$ (i = const). Then the right-hand side of the second expression in (1) will have the form $gi - ku^2/h$. We go over to the variable $\xi = x - wt$ (w = const) in Eqs. (1) and (4) and obtain a system of ordinary differential equations:

$$(u-w)h' + hu' = 0\,,\quad (u-w)u' + gh' + g\tilde{z}_0' = gi - \frac{ku^2}{h}\,,$$

$$-w\tilde{z}_0' + (\alpha - h)u' - uh' = 0\,. \tag{6}$$

Integration of the first expression in (6) gives $h(u - w)$ = const. In the case of a dry channel, const = 0 and $u = w$. Then the other two expressions take the form $gh' + g\tilde{z}_0' = gi - kw^2/h$, $-w\tilde{z}_0' - wh' = 0$. This contradiction shows that there is no wave with a constant velocity over a dry channel. In the case of a wet channel with a uniform flow $h = h_0$, $u = u_0$, and $\tilde{z}_0 = 0$ (here $gih_0 = ku_0^2$) ahead of the break-thhrough wave, the first and third expressions of (6) yield $(u - w)h = (u_0 - w)h_0$, $-w\tilde{z}_0 + (\alpha - h)u = -w\tilde{z}_{00} + (\alpha - h_0)u_0$. Hence it follows that

$$u = w + \frac{(u_0 - w)h_0}{h}\,,\quad \tilde{z}_0 = \frac{1}{w}\left[(\alpha - h)\left(w + \frac{(u_0 - w)h_0}{h}\right) + w\tilde{z}_{00} - (\alpha - h_0)u_0\right].$$

Substituting these values into the second expression of (6), we come to the relation

$$\frac{dh}{d\xi} = \frac{k\,[wh + (u_0 - w)h_0]^2 - gih^3}{h_0(u_0 - w)\left[\dfrac{g\alpha h}{w} + (u_0 - w)h_0\right]}\,.$$

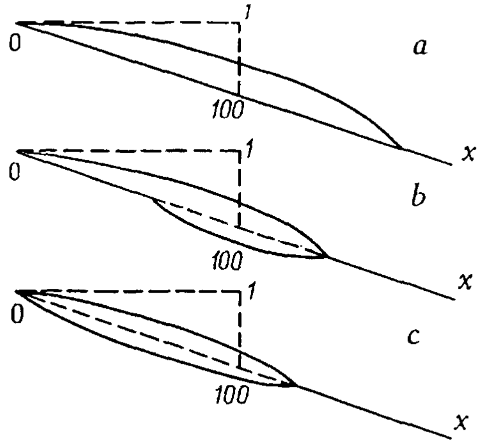

Fig. 1. Distribution of the depth and the marker of the bottom over the break-through wavelength: a) $u_{\text{wash}} = 20$ m/sec; b) 5; c) 0.

The initial condition for this relation follows from the conditions on the discontinuity $h(0) = \bar{h}(w)$. The conditions on the broken wave over the deformable bottom lead to a quadratic equation for h but with unwieldy coefficients. Examination of the last equation gives the leading-edge velocity w and the depth distribution in the vicinity of the leading edge for the case of a wet channel. The solution with a strong discontinuity (the broken wave) is associated with unwieldy formulas and with an investigation of high-power algebraic equations requiring numerical calculations. In the case of a weak discontinuity (discontinuity of the derivatives), when $w = u_0/2 + (u_0^2/4 + g\alpha)^{1/2}$ and $\bar{h} = h_0$, we have the expression

$$\left.\frac{dh}{d\xi}\right|_{h\to h_0} = \frac{2ih_0\left(\frac{u_0}{2} - \sqrt{\left(\frac{u_0^2}{4} + g\alpha\right)}\right)\left(\sqrt{\left(\frac{u_0^2}{4} + g\alpha\right)} - u_0\right)}{\alpha u_0\left(\frac{u_0}{2} - \sqrt{\left(\frac{u_0^2}{4} + g\alpha\right)}\right)}.$$

Hence, requiring that $dh/d\xi < 0$ at $\xi = 0$, we obtain $(u_0^2/4 + g\alpha)^{1/2} - u_0 > 0$ or $u_0 < (4g\alpha/3)^{1/2}$. This is the condition of existence of a leading edge of the break-through wave in the form of a weak discontinuity.

Now we give a variant of the mathematical interpretation of the notion of "washing-out velocity" known in the hydraulic theory of washing-out. The third expression in the system (1) and (4) is represented in the form

$$\frac{\partial z_0}{\partial t} = \begin{cases} 0, & |u| \le u_{\text{wash}}, \\ -(\alpha - h)\dfrac{\partial u}{\partial x} + u\dfrac{\partial h}{\partial x}, & |u| > u_{\text{wash}}. \end{cases} \tag{7}$$

Here we suggest a mechanism of detention of particles of the ground that forms the channel bottom.

It should be noted that at $\partial z_0/\partial t = 0$ we have ordinary St. Venant equations, when $\partial z_0/\partial x$ in the second equation of system (1) is known. Here the characteristics have the form $\lambda_1 = 0$ and $\lambda_{2,3} = u \pm (gh)^{1/2}$. Thus, in passing from the first expression of (7) to the second one and conversely, the characteristics change abruptly, to which a weak discontinuity of the function z_0 corresponds.

Finally, for the system (1) and (7) we set additional conditions in the case of a dry channel. At the leading edge we prescribe the conditions

$$u = w, \quad h = 0, \quad z_0 = z_{00}(x) \quad \text{at} \quad x = l + \int_0^t w(t)\,dt. \tag{8}$$

Conditions (8) are based on an experiment that confirms that the wave front does not wash out the bottom.

At the rear edge of the wave we have

$$u = 0 \, , \quad h = 0 \, , \quad z_0 = z_{00} \, (0) \quad \text{at} \quad x = 0 \, . \tag{9}$$

When $t = 0$, the initial data are prescribed in the form

$$u = u_0 \, (x) \, , \quad h = h_0 \, (x) \, , \quad z_0 = z_{00} \, (x) \quad \text{at} \quad 0 \leq x \leq l \, . \tag{10}$$

The initial boundary-value problem (1), (7)-(10) was solved numerically on a computer. Use was made of a predictor–corrector difference scheme with an explicit Lax diagram on an intermediate layer and recalculation by the "cross" scheme. On the basic layer the equations were taken in divergent form. A moving difference grid tied to the leading edge [1-3] was used. Figure 1 presents results of a numerical calculation of a break-through wave at one and the same instant of time for different values of u_{wash} in Eq. (7). For an arbitrary profile of the bottom, sections of washing-out will be observed. Thus, even a simple model (the ground is homogeneous, i.e., α is constant and u_{wash} is constant) gives physically probable results. Subsequently, more exact but more complex models of washing-out can be developed.

It is of interest to note that liquid flow over a deformable bottom is an example of a physical system with two sharply differing scales of motion: rapid motion for the liquid and slow motion for the ground.

CONCLUSIONS

1. An original derivation of a formula for the flow rate of deposits is presented.
2. A mechanism of detention of ground particles is suggested by introducing the notion of "washing-out velocity."
3. The solution is sought in a time-variable region that is unknown in advance.

NOTATION

u, liquid velocity; h, flow depth; z_0, marker of the bottom; Q, flow rate of deposits; α, constant of the ground; u_{wash}, constant equal to the "washing-out velocity"; g, free-fall acceleration; k, coefficient of hydraulic (turbulent) friction; l, coordinate of the dam; w, leading-edge velocity of the break-through wave; z_{00}, initial marker of the bottom; i, slope of the initial bottom; x, distance; t, time; λ, characteristic; C, A, B, positive constants that depend on u_0, h_0, α, g; τ, ξ, independent variables; $[f]$, discontinuous function; $\tilde{z}_0$, $\tilde{z}_{00}$, certain deviations from the sloping plane; D, discontinuity velocity; m, arbitrary constant; a prime denotes differentiation with respect to ξ.

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